

YOUNG WALLS OF TYPE $D_{n+1}^{(2)}$ AND STRICT PARTITIONS.

SE-JIN OH^{1,2}

ABSTRACT. We show that the number of reduced Young walls of type $D_{n+1}^{(2)}$ with m blocks is independent of n and the same as the number of strict partitions of m . Thus the principally specialized character $\chi_n^{\Lambda_0}(t)$ of $V(\Lambda_0)$ over $U_q(D_{n+1}^{(2)})$ can be interpreted as a generating function for strict partitions. Hence we obtain an infinite family of generalizations of Euler's partition identity.

INTRODUCTION

The characters of integrable modules over quantum groups $U_q(\mathfrak{g})$ are important algebraic invariants which *determine* the isomorphism classes of integrable modules in the sense that $M \cong N$ if and only if $\text{ch} M = \text{ch} N$. In [6, 7], Kashiwara developed the *crystal basis theory* for integrable $U_q(\mathfrak{g})$ -modules from which a lot of combinatorial properties of integrable modules can be deduced. For instance, using an explicit realization of crystal bases, one can compute the characters of integrable modules.

In [4], Kang introduced the notion of *Young walls* as a new combinatorial scheme for realizing the crystal bases of integrable highest weight modules over quantum affine algebras. In that paper, it was shown that the set \mathcal{F} of *proper Young walls* has the crystal structure (induced by the Kashiwara operators \tilde{e}_i, \tilde{f}_i). Moreover, the crystal $B(\Lambda)$ of the basic representation $V(\Lambda)$ was realized as the crystal \mathcal{K} consisting of *reduced Young walls*. Using these realizations, we can derive explicit formulas for the characters of level 1 highest weight modules (see [2, 5] for more details).

A weakly decreasing sequence of non-negative integers $\lambda = (\lambda_1, \lambda_2, \dots)$ is called a *partition* of m , denoted by $\lambda \vdash m$, if $|\lambda| := \sum_i \lambda_i = m$. A partition λ is called a *strict partition* if all the nonzero parts are strictly decreasing and an *odd partition* if all the nonzero parts are odd. Let $\mathcal{R}(m)$ (respectively, $\mathcal{Q}(m)$) be the number of strict (respectively, odd) partitions of m . Then *Euler's partition identity* states that $\mathcal{R}(m) = \mathcal{Q}(m)$ because the generating function for strict partitions and the one for odd partitions are the same (see [1], for more details):

$$\prod_{i=1}^{\infty} (1 + t^i) = \prod_{i=1}^{\infty} \frac{1}{1 - t^{2i-1}}.$$

In this paper, we show that the *principally specialized character* $\chi_n^{\Lambda_0}(t)$ of the basic representation $V(\Lambda_0)$ over $U_q(D_{n+1}^{(2)})$ can be interpreted as a generating function for strict partitions. More precisely,

2000 *Mathematics Subject Classification.* 81R50, 17B37, 17B65, 05A17.

Key words and phrases. crystal basis, Euler's partition theorem, strict partition, Young walls.

¹ This work was supported by BK21 Mathematical Sciences Division.

² This work was supported by NRF Grant # 2010-0010753.

the number of reduced Young walls of type $D_{n+1}^{(2)}$ with m blocks, $\mathcal{R}(m)$, coincides with the number of strict partitions of m . In particular, it does *not* depend on n , which is a rather surprising fact. (This fact was already discovered in [10] using the technique of vertex operators.) Thus we obtain infinite families of partitions for which the Euler's partition identity hold. Furthermore, by defining the notion of a *virtual character* for strict partitions, we show that the number of strict partitions of weight $\Lambda_0 - \alpha$ is the same as the number of reduced Young walls of weight $\Lambda_0 - \alpha$, which leads the the following conjecture.

Conjecture 0.1. *The set \mathcal{S} of strict partitions has a $U_q(D_{n+1}^{(2)})$ -crystal structure and it is isomorphic to the highest weight crystal $B(\Lambda_0)$ over $U_q(D_{n+1}^{(2)})$ for every $n \in \mathbb{Z}_{\geq 2}$.*

Acknowledgements. The author would like to express his sincere gratitude to Research Institute for Mathematical Sciences, Kyoto University for their hospitality during his visit in January, 2011. After the author was finished with this paper, he was informed by Shunsuke Tsuchioka of the result in [10]. The author is very grateful to him for pointing it out. The author would also like to thank Prof. Seok-Jin Kang and Daehong Kim for many valuable discussions and suggestions.

1. THE QUANTUM AFFINE ALGEBRA $U_q(D_{n+1}^{(2)})$

Let $I = \{0, 1, \dots, n\}$ ($n \geq 2$) be the index set. The *affine Cartan datum* $(A, P^\vee, P, \Pi^\vee, \Pi)$ of type $D_{n+1}^{(2)}$ consists of

(1) the *Cartan matrix*

$$A = (a_{ij})_{i,j \in I} = \begin{pmatrix} 2 & -1 & 0 & \cdots & \cdots & 0 \\ -2 & 2 & -1 & \cdots & \cdots & 0 \\ 0 & -1 & 2 & \ddots & \cdots & 0 \\ \vdots & \cdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \cdots & \cdots & -1 & 2 & -2 \\ 0 & \cdots & \cdots & 0 & -1 & 2 \end{pmatrix}$$

(2) a free abelian group $P^\vee = \bigoplus_{i=0}^n \mathbb{Z}h_i \oplus \mathbb{Z}d$, the *dual weight lattice*,

(3) a free abelian group $P = \bigoplus_{i=0}^n \mathbb{Z}\Lambda_i \oplus \mathbb{Z}\delta \subset \mathfrak{h}^* = \mathbb{C} \otimes_{\mathbb{Z}} P^\vee$, the *weight lattice*,

(4) $\Pi^\vee = \{h_i \mid i \in I\} \subset P^\vee := \text{Hom}(P, \mathbb{Z})$, the *set of simple coroots*,

(5) $\Pi = \{\alpha_i \in P \mid i \in I\}$, the set of *simple roots*,

satisfying the following properties:

(a) $\langle h_i, \alpha_j \rangle = a_{ij}$ for all $i, j \in I$ and $\langle d, \alpha_j \rangle = \delta_{j0}$,

(b) Π is linearly independent,

(c) $\langle h_j, \Lambda_i \rangle = \delta_{ij}$ for all $j \in I$, $\langle d, \Lambda_i \rangle = 0$,

(d) $\langle h_j, \delta \rangle = 0$ and $\langle d, \delta \rangle = 1$.

We denote by $P^+ := \{\Lambda \in P \mid \langle h_i, \Lambda \rangle \in \mathbb{Z}_{\geq 0}, i \in I\}$ the set of *dominant integral weights*. The free abelian group $Q := \sum_{i \in I} \mathbb{Z}\alpha_i$ is called the *root lattice* and we denote by $Q^+ := \bigoplus_{i \in I} \mathbb{Z}_{\geq 0}\alpha_i$. For $\alpha = \sum_{i \in I} k_i \alpha_i \in Q$, we define the *height* of α to be $\text{ht}(\alpha) := \sum_{i \in I} k_i$. Note that the Cartan matrix is *symmetrizable*; i.e., there is a diagonal matrix $D = \text{diag}(1, 2, \dots, 2, 1)$ such that DA is symmetric.

Let q be an indeterminate. For $i \in I$ and $m, n \in \mathbb{Z}_{\geq 0}$, define

$$q_0 = q_n = q, \quad q_1 = \dots = q_{n-1} = q^2, \quad [n]_{q_i} = \frac{q_i^n - q_i^{-n}}{q_i - q_i^{-1}}, \quad [n]_{q_i}! = \prod_{k=1}^n [k]_{q_i}, \quad \begin{bmatrix} m \\ n \end{bmatrix}_{q_i} = \frac{[m]_{q_i}!}{[m-n]_{q_i}! [n]_{q_i}!}.$$

Definition 1.1. The *quantum group* $U_q(D_{n+1}^{(2)})$ with a Cartan datum $(A, P^\vee, P, \Pi^\vee, \Pi)$ is the associative algebra over $\mathbb{C}(q)$ with **1** generated by e_i, f_i ($i \in I$) and q^h ($h \in P^\vee$) satisfying the following relations:

- (1) $q^0 = 1, q^h q^{h'} = q^{h+h'}$ for $h, h' \in P^\vee$,
- (2) $q^h e_i q^{-h} = q^{\langle h, \alpha_i \rangle} e_i, q^h f_i q^{-h} = q^{-\langle h, \alpha_i \rangle} f_i$ for $h \in P^\vee, i \in I$,
- (3) $e_i f_j - f_j e_i = \delta_{ij} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}}$, where $K_i = q_i^{h_i}$,
- (4) $\sum_{k=0}^{1-a_{ij}} \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_{q_i} e_i^{1-a_{ij}-k} e_j e_i^k = 0$ if $i \neq j$,
- (5) $\sum_{k=0}^{1-a_{ij}} \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_{q_i} f_i^{1-a_{ij}-k} f_j f_i^k = 0$ if $i \neq j$.

A $U_q(\mathfrak{g})$ -module V is called a *weight module* if it admits a *weight space decomposition* $V = \bigoplus_{\mu \in P} V_\mu$, where $V_\mu = \{v \in V \mid q^h v = q^{\langle h, \mu \rangle} v \text{ for all } h \in P^\vee\}$. If $\dim_{\mathbb{C}(q)} V_\mu < \infty$ for all $\mu \in P$, we define the *character* of V by

$$\chi_{D_{n+1}^{(2)}}(V) = \sum_{\mu \in P} (\dim_{\mathbb{C}(q)} V_\mu) e(\mu),$$

where $e(\mu)$ is an basis element of the group algebra $\mathbb{Z}[P]$ with the multiplication given by $e(\mu)e(\nu) = e(\mu + \nu)$ for all $\mu, \nu \in P$.

Then it is proved in [9], [2, Chapter 3] that the category of integrable modules is semisimple with its irreducible objects being isomorphic to $V_n(\Lambda)$ for some $\Lambda \in P^+$ such that

- it is generated by a unique highest weight v_Λ ,
- q^h acts on v_Λ by a multiplication of $q^{\langle h_i, \Lambda \rangle}$ for all $h \in P^\vee$,
- e_i and $f_i^{\langle h_i, \Lambda \rangle + 1}$ act trivially on v_Λ for all $i \in I$,
- it admits a weight space decomposition, $V_n(\Lambda) = \bigoplus_{\mu \in P} V_n(\lambda)_\mu$.

For $V_n(\Lambda)$ ($\Lambda \in P^+$), we set $\chi_n^\Lambda = \chi_{D_{n+1}^{(2)}}(V_n(\Lambda))$, when there is no danger of confusion.

Definition 1.2. We define the *principally specialized character* of $V_n(\Lambda)$ as follows:

$$\chi_{D_{n+1}^{(2)}}^\Lambda(t) = \chi_n^\Lambda(t) = \sum_m \left(\sum_{\substack{\mu \in P \\ \text{ht}(\Lambda - \mu) = m}} \dim V_n(\Lambda)_\mu \right) t^m,$$

where t is the indeterminate.

Note that $\chi_n^\Lambda(t)$ can be derived by specializing $e(\Lambda) = 1$, $e(-\alpha_i) = t$ ($i \in I$) in χ_n^Λ ; i.e.,

$$\chi_n^\Lambda(t) = \chi_n^\Lambda|_{\substack{e(\Lambda)=1 \\ e(-\alpha_i)=t}}, \text{ for all } i \in I.$$

The *level* of $\Lambda \in P^+$ is defined to be the nonnegative integer $\langle c, \Lambda \rangle$, where $c = h_0 + 2h_1 + \cdots + 2h_{n-1} + h_n$. Thus the dominant integral weights of level 1 are Λ_0 and Λ_n .

Let $\mathbb{A}_0 = \{f/g \in \mathbb{C}(q) \mid f, g \in \mathbb{C}[q], g(0) \neq 0\}$. It is shown in [7] that $V_n(\Lambda)$ has a unique *crystal basis* $(L(\Lambda), B(\Lambda))$, where $L(\Lambda)$ is a free \mathbb{A}_0 -lattice of $V_n(\Lambda)$, $B(\Lambda)$ is a \mathbb{C} -basis of $L(\Lambda)/qL(\Lambda)$ and $B(\Lambda)$ has a I -colored oriented graph structure induced by the *Kashiwara* operators \tilde{e}_i, \tilde{f}_i ($i \in I$). Moreover, $B(\Lambda)$ encodes the combinatorial information of $V_n(\Lambda)$ as follows:

- $B_n(\Lambda)$ admits a weight space decomposition. i.e,

$$B_n(\Lambda) = \bigsqcup_{\mu \in P} B_n(\Lambda)_\mu \text{ where } B_n(\Lambda)_\mu = B_n(\Lambda) \cap V_n(\Lambda)_\mu,$$

- $|B_n(\Lambda)_\mu| = \dim_{\mathbb{C}(q)} V_n(\Lambda)_\mu = \dim_{\mathbb{C}} V_n(\Lambda)_\mu$.

Thus χ_n^Λ and $\chi_n^\Lambda(t)$ can be expressed as follows:



$$(1.1) \quad \chi_n^\Lambda = \sum_{\mu \in P} |B_n(\Lambda)_\mu| e(\mu), \quad \chi_n^\Lambda(t) = \sum_m \left(\sum_{\substack{\mu \in P \\ \text{ht}(\Lambda - \mu) = m}} |B_n(\Lambda)_\mu| \right) t^m.$$

In particular, $B(\Lambda)$ becomes a $U_q(D_{n+1}^{(2)})$ -crystal. (See [2, 8] for more details on the crystal bases theory.)

2. YOUNG WALL REALIZATION OF $B_n(\Lambda)$ OF LEVEL 1

In [4], Kang gave a realization of level 1 highest weight crystals $B(\Lambda)$ for all classical quantum affine algebras in terms of *reduced Young walls* (see [3] as well). Hereafter, we briefly introduce the result in [4] only for the type $D_{n+1}^{(2)}$. Since $V_n(\Lambda_0)$ can be identified with $V_n(\Lambda_n)$ by symmetry, we will consider the case of Λ_0 only.

Basically, the Young walls are built of colored blocks. In case of the type $D_{n+1}^{(2)}$, there are two types of blocks whose shapes are different as follows:

- Unit block  whose colors are $1, \dots, n-1$,
- Half-height block  whose colors are $0, n$.

Then the set of Young walls is the set of blocks built on the *ground-state Young wall* by the following rules:

- All blocks should be placed on top of the ground-state Young wall or another block.
- The colored blocks should be stacked in the pattern given below.
- Except for the right-most column, there should be no free space to the right of any blocks.

The *ground-state Young wall* Y_{Λ_0} and the pattern in (b) are given by follows:

...	1	1	1	1	1
...	0	0	0	0	0
...	0	0	0	0	0
...	1	1	1	1	1
...	:	:	:	:	:
...	$n-1$	$n-1$	$n-1$	$n-1$	$n-1$
...	n	n	n	n	n
...	n	n	n	n	n
...	$n-1$	$n-1$	$n-1$	$n-1$	$n-1$
...	:	:	:	:	:
...	1	1	1	1	1
...	0	0	0	0	0
...	0	0	0	0	0

where the blocks in bottom are the ground-state Young wall Y_{Λ_0} .

A column in Young wall is called a *full column* if its height is a multiple of unit length. We say a Young wall *proper* if none of the full columns have same height. The part of a column consisting of 2 0-blocks, 2 1-blocks, \dots , 2 n -blocks is called a δ -column.

Definition 2.1.

- (1) A column in a proper Young wall is said to contain a *removable δ* if we may remove a δ -column from Y and still obtain a proper Young wall.
- (2) A proper Young wall is said to be *reduced* if none of its columns contain a removable δ .

For a given Young wall Y , we define the *weight*, $\text{wt}(Y)$, of Y as follows:

$$(2.1) \quad \text{wt}(Y) = \Lambda_0 - \sum_{i \in I} a_i \alpha_i,$$

where a_i is the number of i -blocks on the ground-state Young wall Y_{Λ_0} .

Let \mathcal{F}_n be the set of all proper Young walls built on Y_{Λ_0} and \mathcal{K}_n be the set of all reduced proper Young walls built on Y_{Λ_0} .

Theorem 2.2. [4]

- (1) \mathcal{F}_n has a crystal structure induced by the Kashiwara operators \tilde{e}_i and \tilde{f}_i .
- (2) There is a crystal isomorphism between \mathcal{K}_n and $B_n(\Lambda)$.

Definition 2.3. For a $m \in \mathbb{Z}_{\geq 0}$, $\mu \in P$ and subset \mathcal{A} of \mathcal{F}_n , we define

- (1) $\mathcal{A}[m]$ to be the subset of \mathcal{A} which has m blocks on Y_{Λ_0} ,
- (2) $\mathcal{A}[\mu]$ to be the subset of \mathcal{A} consisting of Young walls with weight μ ,

(3) the *virtual character* $\overset{\circ}{\chi}_n(\mathcal{A})$ of \mathcal{A} to be

$$\overset{\circ}{\chi}_n(\mathcal{A}) = \sum_{\mu \in P} |\mathcal{A}[\mu]| e(\mu).$$

Then the equations in (1.1) can be written by

$$(2.2) \quad \chi_n^{\Lambda_0} = \overset{\circ}{\chi}_n(\mathcal{K}_n) = \sum_{\mu \in P} |\mathcal{K}_n[\mu]| e(\mu), \quad \chi_n^{\Lambda_0}(t) = \sum_m |\mathcal{K}_n[m]| t^m.$$

Proposition 2.4. [5, Corollary 2.5]

$$\mathcal{F}_n = \bigoplus_{k \in \mathbb{Z}_{\geq 0}} B_n(\Lambda_0 - 2k(\sum_{i=0}^n \alpha_i))^{\oplus \mathcal{P}(k)},$$

where $\mathcal{P}(k)$ is the number of partitions of k .

In terms of Young walls, Proposition 2.4 can be interpreted as follows:

$$(2.3) \quad |\mathcal{F}_n[m]| = \sum_{\substack{k \geq 0 \\ m - 2(n+1)k \geq 0}} (|\mathcal{K}_n[m - 2(n+1)k]| \times \mathcal{P}(k)).$$

3. PRINCIPALLY SPECIALIZED CHARACTER OF $V_n(\Lambda_0)$

Fix $\Delta = n + 1$. For a given proper Young wall $Y \in \mathcal{F}_n$, we can associate a partition $\lambda^Y = (\lambda_1^Y, \dots, \lambda_m^Y, \dots) \vdash |Y|$, where λ_i^Y is the number of blocks in i th-column above the ground-state wall and $|Y| = \sum_{k \in \mathbb{Z}_{\geq 0}} \lambda_k^Y$. Thus \mathcal{F}_n and \mathcal{K}_n can be expressed as the sets of partitions as follows:

$$\mathcal{F}_n = \{\lambda = (\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_m, \dots) \mid \lambda_i = \lambda_{i+1} \text{ implies } \lambda_i = t\Delta \text{ for some } t \in \mathbb{Z}_{\geq 0}\}.$$

$$\mathcal{K}_n = \{\lambda \in \mathcal{F}_n \mid \lambda_i - \lambda_{i+1} \leq 2\Delta, \text{ with equality only if } \lambda_i \neq t\Delta \text{ for all } t \in \mathbb{Z}_{\geq 0}\}.$$

Let \mathcal{S}_n be the subset of \mathcal{F}_n consisting of strictly decreasing sequences of non-negative integers; i.e.,

$$\mathcal{S}_n = \{\lambda \in \mathcal{F}_n \mid \lambda_i = \lambda_{i+1} \text{ implies } \lambda_i = 0\}.$$

From Definition 2.3,

$$(3.1) \quad \overset{\circ}{\chi}_n(\mathcal{S}_n[m]) = \sum_{\mu \in P} |\mathcal{S}_n[m][\mu]| e(\mu), \quad \overset{\circ}{\chi}_n(\mathcal{S}_n) = \sum_{\mu \in P} |\mathcal{S}_n[\mu]| e(\mu).$$

Since the set of strict partitions \mathcal{S}_n does not depend on n , we will drop the subindex n when we want to emphasize the independence.

Example 3.1. For $S[7]$,

$$\overset{\circ}{\chi}_2(S[7]) = 3e(\Lambda_0 - (3\alpha_0 + 2\alpha_1 + 2\alpha_2)) + e(\Lambda_0 - (2\alpha_0 + 2\alpha_1 + 3\alpha_2)) + e(\Lambda_0 - (2\alpha_0 + 3\alpha_1 + 2\alpha_2)),$$

$$\begin{aligned} \overset{\circ}{\chi}_3(S[7]) = & e(\Lambda_0 - (3\alpha_0 + 2\alpha_1 + \alpha_2 + \alpha_3)) + e(\Lambda_0 - (2\alpha_0 + 2\alpha_1 + 2\alpha_2 + \alpha_3)) + e(\Lambda_0 - (2\alpha_0 + 2\alpha_1 + \alpha_2 + 2\alpha_3)) + \\ & e(\Lambda_0 - (2\alpha_0 + \alpha_1 + 2\alpha_2 + 2\alpha_3)) + e(\Lambda_0 - (\alpha_0 + 2\alpha_1 + 2\alpha_2 + 2\alpha_3)). \end{aligned}$$

We will denote by \mathcal{K}_n^c (respectively, \mathcal{S}_n^c) the complement of \mathcal{K}_n (respectively, \mathcal{S}_n) in \mathcal{F}_n . Note that the set \mathcal{S}_n^c depends on n . However, the following theorem tells us that the cardinality of \mathcal{S}_n^c does not depend on n .

Theorem 3.2. *There are 1 – 1 and onto maps given as follows:*

$$(1) \quad \overline{\Psi}_n^m : \mathcal{K}_n^c[m] \rightarrow \bigsqcup_{\substack{k>0 \\ m-2k\Delta \geq 0}} (\mathcal{K}_n[m-2k\Delta] \times \{\lambda \vdash k\}).$$

$$(2) \quad \overline{\Phi}_n^m : \mathcal{S}_n^c[m] \rightarrow \bigsqcup_{\substack{k>0 \\ m-2k\Delta \geq 0}} (\mathcal{S}[m-2k\Delta] \times \{\lambda \vdash k\}).$$

Proof. (1) Actually, the first assertion comes from (2.3), directly. In this proof, we will give an explicit 1 – 1 and onto map between $\mathcal{K}_n^c[m]$ and $\bigsqcup_{\substack{k>0 \\ m-2k\Delta \geq 0}} (\mathcal{K}_n[m-2k\Delta] \times \{\lambda \vdash k\})$. Let Ψ_n^m be a map from

$\mathcal{K}_n^c[m]$ to $\bigsqcup_{\substack{k>0 \\ m-2k\Delta \geq 0}} \mathcal{K}_n[m-2k\Delta]$ by the following algorithm **A**

(A1) Let $\lambda \in \mathcal{K}_n^c[m]$ be given. Set $\lambda^{(0)} = \lambda$ and $l = 0$.

(A2) Find maximal i such that

$$(3.2) \quad \lambda_{i-1}^{(l)} - \lambda_i^{(l)} \geq 2t\Delta \text{ for some } t \in \mathbb{Z}_{>0} \text{ with equality hold only if } \lambda_{i-1}^{(l)} = k\Delta \text{ for some } k.$$

(A3) Among t 's satisfying the inequality in (3.2), choose the maximal one and say t_l . Set

$$\lambda^{(l+1)} := (\lambda_1^{(l)} - 2t_l\Delta, \lambda_2^{(l)} - 2t_l\Delta, \dots, \lambda_{i-1}^{(l)} - 2t_l\Delta, \lambda_i^{(l)}, \lambda_{i+1}^{(l)}, \dots).$$

(A4) If there is no i such that

$$\lambda_{i-1}^{(l+1)} - \lambda_i^{(l+1)} \geq 2t\Delta \text{ for some } t \in \mathbb{Z}_{>0} \text{ and equality holds for } \lambda_{i-1}^{(l+1)} = k\Delta \text{ for some } k,$$

define $\bar{\lambda} = \lambda^{(l+1)}$ and terminate algorithm, otherwise $l = l + 1$ and go to (A2).

Then this algorithm terminates in a finite step and one can check the following things:

- $k = \frac{|\lambda| - |\bar{\lambda}|}{2\Delta} \in \mathbb{Z}_{>0}$,
- $\bar{\lambda} \in \mathcal{K}_n[m-2k\Delta]$,
- $\hat{\lambda}_i := \frac{\lambda_i - \bar{\lambda}_i}{2\Delta} \in \mathbb{Z}_{\geq 0}$,
- $\hat{\lambda} := (\hat{\lambda}_1, \hat{\lambda}_2, \dots) \vdash k$.

Thus we can get a function

$$\overline{\Psi}_n^m : \mathcal{K}_n^c[m] \rightarrow \bigsqcup_{\substack{k>0 \\ m-2k\Delta \geq 0}} (\mathcal{K}_n[m-2k\Delta] \times \{\lambda \vdash k\})$$

given by $\lambda \mapsto (\bar{\lambda}, \hat{\lambda})$. Then one can show that $\overline{\Psi}_n^m$ is a 1 – 1 and onto map with corresponding preimage of $(\bar{\lambda}', \hat{\lambda}') \in \mathcal{K}_n[m-2k\Delta] \times \{\lambda \vdash k\}$ given by

$$\lambda' := (\bar{\lambda}'_1 + 2\hat{\lambda}'_1\Delta, \bar{\lambda}'_2 + 2\hat{\lambda}'_2\Delta, \dots, \bar{\lambda}'_k + 2\hat{\lambda}'_k\Delta, \dots).$$

(2) For a given partition λ , define the inserting $\underbrace{(k\Delta, \dots, k\Delta)}_j$ into λ denoted by $(k\Delta)^j \hookrightarrow \lambda$ as follows:

- Find i such that $\lambda_i \geq k\Delta > \lambda_{i+1}$,
- Set $(k\Delta)^j \hookrightarrow \lambda := (\lambda_1, \lambda_2, \dots, \lambda_i, \underbrace{k\Delta, \dots, k\Delta}_j, \lambda_{i+1}, \dots)$.

Let Φ_n^m be a map from $\mathcal{S}_n^c[m]$ to $\bigsqcup_{\substack{k>0 \\ m-2k\Delta \geq 0}} \mathcal{S}[m-2k\Delta]$ by the following algorithm **B**

(B1) Let $\lambda \in \mathcal{S}_n^c[m]$ be given. Set $\lambda^{(0)} = \lambda$ and $l = 0$.

(B2) Find maximal i such that

$$\lambda_{i-1}^{(l)} = \lambda_i^{(l)} = \widehat{\lambda}_l \Delta \text{ for some } \widehat{\lambda}_l \in \mathbb{Z}_{>0}.$$

(B3) Set

$$\lambda^{(l+1)} := (\lambda_1^{(l)}, \lambda_2^{(l)}, \dots, \lambda_{i-2}^{(l)}, \lambda_{i+1}^{(l)}, \dots).$$

(B4) If there is no j such that

$$\lambda_{j-1}^{(l+1)} = \lambda_j^{(l+1)} = k\Delta \text{ for some } k \in \mathbb{Z}_{>0},$$

define $\overline{\lambda} = \lambda^{(l+1)}$ and terminate algorithm, otherwise $l = l + 1$ and go to (B2).

Then this algorithm terminates in a finite step and one can check the following things:

- $k = \frac{|\lambda| - |\overline{\lambda}|}{2\Delta} \in \mathbb{Z}_{>0}$,
- $\overline{\lambda} \in \mathcal{S}[m - 2k\Delta]$,
- $\widehat{\lambda} := (\widehat{\lambda}_l, \widehat{\lambda}_{l-1}, \dots, \widehat{\lambda}_1) \vdash k$.

Thus we can get a function

$$\overline{\Phi}_n^m : \mathcal{S}_n^c[m] \rightarrow \bigsqcup_{\substack{k>0 \\ m-2k\Delta \geq 0}} (\mathcal{S}[m-2k\Delta] \times \{\lambda \vdash k\})$$

given by $\lambda \mapsto (\overline{\lambda}, \widehat{\lambda})$. Then one can show that $\overline{\Phi}_n^m$ is an 1-1 and onto map with corresponding preimage of $(\overline{\lambda}', \widehat{\lambda}') \in \mathcal{S}[m-2k\Delta] \times \{\lambda \vdash k\}$ given by

$$\lambda' := (\widehat{\lambda}'_l \Delta)^2 \hookrightarrow (\widehat{\lambda}'_{l-1} \Delta)^2 \hookrightarrow \dots \hookrightarrow (\widehat{\lambda}'_1 \Delta)^2 \hookrightarrow \overline{\lambda}'.$$

□

Corollary 3.3. For all $m \in \mathbb{Z}_{\geq 0}$ and $n \in \mathbb{Z}_{\geq 1}$

$$|\mathcal{K}_n[m]| = |\mathcal{S}[m]|.$$

Proof. One can show that $\mathcal{S}[t] = \mathcal{K}_n[t]$ for $0 \leq t \leq 2\Delta$. The Theorem 3.2 tells us that for all $m > 2\Delta$, $|\mathcal{S}_n^c[m]|$ depends on the set of strict partitions of $k \in \mathbb{Z}_{\geq 0}$ and $\mathcal{P}(k)$ such that $\frac{m-k}{2\Delta} \in \mathbb{Z}_{>0}$. Then by using induction on m , we can conclude that $|\mathcal{S}_n^c[m]| = |\mathcal{K}_n^c[m]|$. Hence

$$|\mathcal{S}[m]| = |\mathcal{K}_n[m]|.$$

Remark 3.6. For an affine type $A_n^{(1)}$, the combinatorial realizations of crystal bases of level 1 are well-known as $n+1$ -reduced colored Young diagrams. One can check that the set of 2-reduced Young diagrams is identified with the set of strict partitions by transposing diagrams. Hence the principal specialized character of irreducible modules of level 1 over $A_1^{(1)}$ satisfies the Euler's partition identity:

$$(3.3) \quad \chi_{A_1^{(1)}}^\Lambda(t) = \prod_{i=1}^{\infty} (1 + t^i) = \prod_{i=1}^{\infty} \frac{1}{1 - t^{2i-1}}.$$

The Dynkin diagram of $A_n^{(1)}$ and $D_{n+1}^{(2)}$ are given as follows:

$$\begin{array}{ll} \begin{array}{c} \circ \longleftrightarrow \circ \\ 0 \qquad 1 \end{array} & \text{for } A_1^{(1)}, \\ \begin{array}{c} \circ \longleftrightarrow \circ \longleftrightarrow \circ \\ 0 \qquad 1 \qquad 2 \end{array} & \text{for } D_3^{(2)}, \\ \begin{array}{c} \circ \longleftrightarrow \circ \text{---} \circ \longleftrightarrow \circ \\ 0 \qquad 1 \qquad 2 \qquad 3 \end{array} & \text{for } D_4^{(2)}, \\ \begin{array}{c} \circ \longleftrightarrow \circ \text{---} \cdots \text{---} \circ \longleftrightarrow \circ \\ 0 \qquad 1 \qquad \qquad n-1 \qquad n \end{array} & \text{for } D_{n+1}^{(2)} \ (n \geq 4). \end{array}$$

Thus our result can be interpreted as generalization of the Euler's partition identity in the sense that the leftmost term in equation (3.3) can be replaced to the $\chi_{D_{n+1}^{(2)}}^{\Lambda_0}(t)$, for all $n \in \mathbb{Z}_{\geq 2}$.

From now on, we show that the equality in Corollary 3.3 can be interpreted in more stronger sense. Then we can explain the reason why we have Conjecture 0.1.

In Theorem 3.2, we defined the maps Φ_n^m and Ψ_n^m . By their constructions, we can observe that for $\lambda \in \mathcal{F}_n[m]$,

$$(3.4) \quad \text{wt}(\Psi_n^m(\lambda)) = \text{wt}(\Phi_n^m(\lambda)) = \text{wt}(\lambda) + 2k \left(\sum_{i=0}^n \alpha_i \right), \text{ for some } k \in \mathbb{Z}_{\geq 0}.$$

Theorem 3.7. For all $m \in \mathbb{Z}_{\geq 0}$ and $n \in \mathbb{Z}_{\geq 2}$,

$$\mathring{\chi}_n(\mathcal{S}[m]) = \mathring{\chi}_n(\mathcal{K}_n[m]).$$

Proof. By the definition, $\mathring{\chi}(\mathcal{S}[t]) = \mathring{\chi}(\mathcal{K}_n[t])$ for $0 \leq t < 2\Delta$. For $t = 2\Delta$, one can easily check that $\mathcal{S}[t] \setminus \{(2\Delta, 0, \dots)\} = \mathcal{K}_n[t] \setminus \{(\Delta, \Delta, 0, \dots)\}$. Hence the equation holds for $0 \leq t \leq 2\Delta$. From Theorem 3.2 and the equation (3.4), for $m > 2\Delta$, the $\mathring{\chi}(\mathcal{S}_n^c[m])$ depends on $k \in \mathbb{Z}_{\geq 0}$, the set of strict partitions of k and $\mathcal{P}(k)$ such that $\frac{m-k}{2\Delta} \in \mathbb{Z}_{\geq 0}$. As in the similar way of proof of Corollary 3.3, $\mathring{\chi}(\mathcal{S}_n^c[m]) = \mathring{\chi}(\mathcal{K}_n^c[m])$ and hence our assertion holds. \square

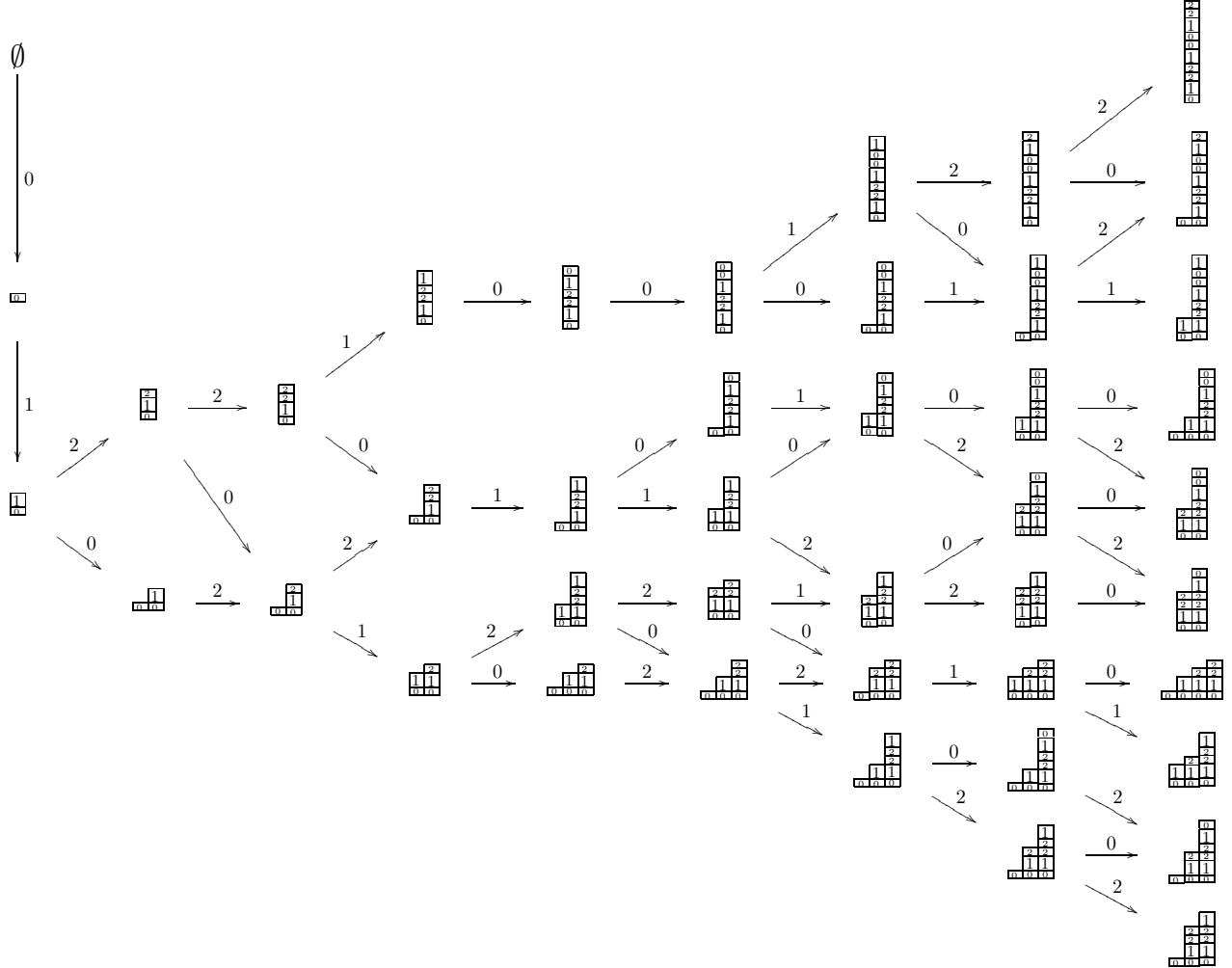
Thus, we conclude the following:

Corollary 3.8. For all $n \in \mathbb{Z}_{\geq 2}$,

$$\chi_n^{\Lambda_0} = \mathring{\chi}_n(\mathcal{S}).$$

From Theorem 3.7 and Corollary 3.8, one may conjecture that there are crystal structures of $B_n(\Lambda_0)$ on the set of strict partitions \mathcal{S} for all $n \in \mathbb{Z}_{\geq 2}$. For example, our conjectured crystal structure on \mathcal{S} of type $D_3^{(2)}$ is given by follows:

Example 3.9.



REFERENCES

1. G. E. Andrews, *On generalizations of Eulers partition theorem*, Mich. Math. J. **13** (1966) 491–498.
2. J. Hong, S.-J. Kang, *Introduction to Quantum Groups and Crystal Bases*, Grad. Stud. Mathematics **42**, Amer. Math. Soc., 2002.
3. Jin Hong, Seok-Jin Kang, Hyeonmi Lee, *Young wall realization of crystal graphs for $U_q(C_n^{(1)})$* , Comm. Math. Phys **244** (2004), 111–131.

4. S.-J. Kang, *Crystal bases for quantum affine algebras and combinatorics of Young walls*, Proc. London Math. Soc. (3) **86** (2003), no.1, 29–69 .
5. S.-J. Kang, J.-H. Kwon, *Crystal bases of the Fock space representations and string functions*, J. Algebra **280** (2004), 313–349
6. M. Kashiwara, *Crystalizing the q -analogue of universal enveloping algebras*, Commun. Math. Phys. **133** (1990), 249–260.
7. ———, *On crystal bases of the q -analogue of universal enveloping algebras*, Duke Math. J. **63** (1991), 465–516.
8. ———, *Global crystal bases of quantum groups*, Duke Math. J. **69** (1993), no. 2, 455–485.
9. G. Lusztig, *Introduction to Quantum Groups*, Progress in Mathematics, 110, Birkhäuser Boston, Inc., Boston, MA, 1993.
10. T. Nakajima, H.-F. Yamada, *Basic representations of $A_{2l}^{(2)}$ and $D^{(2)}_{l+1}$ and the polynomial solutions to the reduced BKP hierarchies*, J. Phys. A **27** (6) (1994) 171–176.

DEPARTMENT OF MATHEMATICAL SCIENCES, SEOUL NATIONAL UNIVERSITY, 599 GWANAK-RO, GWANAK-GU, SEOUL 151-747, KOREA

E-mail address: sj092@snu.ac.kr